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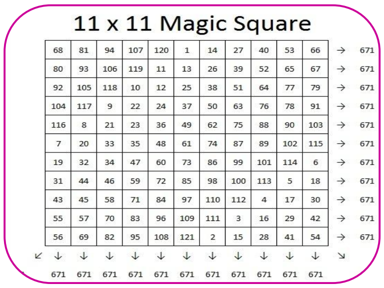
REVIEW OF LITERATURE

PSEUDO MAGIC ON SQUARES OF SQUARES


11 x 11 Magic Square

68	81	94	107	120	1	14	27	40	53	66	→	671
80	93	106	119	11	13	26	39	52	65	67	→	671
92	105	118	10	12	25	38	51	64	77	79	→	671
104	117	9	22	24	37	50	63	76	78	91	→	671
116	8	21	23	36	49	62	75	88	90	103	→	671
7	20	33	35	48	61	74	87	89	102	115	→	671
19	32	34	47	60	73	86	99	101	114	6	→	671
31	44	46	59	72	85	98	100	113	5	18	→	671
43	45	58	71	84	97	110	112	4	17	30	→	671
55	57	70	83	96	109	111	3	16	29	42	→	671
56	69	82	95	108	121	2	15	28	41	54	→	671
↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↘	
671	671	671	671	671	671	671	671	671	671	671		

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ABSTRACT:

This paper is concerned with the problem of finding a 3 x 3 magic square all of whose entries are perfect squares. The paper describes elementary methods of obtaining infinitely many parametric solutions of 3 x 3 pseudo-magic squares of squares, that is, squares constructed from 9 integer squares in which the entries in all the rows and columns as well as one of the diagonals add up to the same magic sum. The problem of finding a 3 x 3 magic square all of whose entries are perfect squares remains open.

KEY WORDS: *squares, squares of squares & parametric solutions*

1. INTRODUCTION :

A magic square is a square array of a2 distinct positive integers such that the sums of the a integers in each row and column and each of the two diagonals are all equal. A square that fails to be magic only because one or both of the diagonal sums differs from the common sum is called a semimagic square. A semimagic square in which the entries in one of the diagonals also add up to the common sum will be called a pseudo- magic square. The common sum of the rows and columns of a semimagic, pseudo-magic or magic square is called the magic sum.

Magic squares have attracted the attention of both amateurs and mathematicians for centuries. A detailed exposition of the theory on the subject is given by Andrews [1] and Kraitchik [8, Chapter 7, pp. 142—192]. Despite the long history of the subject, there are several open problems concerning magic squares. One such problem, posed by LaBar [9] and again mentioned in ([6, p. 269], [7]) is to prove or disprove that a 3 x 3 magic square can be constructed from nine distinct integer squares. Sallows [12] and Schweitzer (as quoted in [6, p. 269]) found numerical examples of pseudo-magic squares of perfect squares. Bremner [3, 4] has studied the problem using technique of algebraic geometry, and has obtained numerous parametric solutions of pseudo-magic squares of squares. The simplest solution found by Bremner has entries of degree 8 in terms of a single parameter. LaBar's problem has also been considered in [2, 10, 11].

As all the earlier known examples of pseudo-magic squares of squares had a perfect square as their magic sum, Guy [6, p. 269] asked the question "Does this have to happen?". A solitary numerical counterexample to this question, attributed to Schweitzer, is given in [2]. In this paper, we first obtain in Section 2 all semimagic squares whose entries are perfect squares. We then give in Section 3.1 an elementary method of obtaining infinitely many pseudo-magic squares of perfect squares, in parametric form, in which the magic sum is, in general, not a perfect square. We also give certain additional numerical examples of pseudo-magic squares of squares that are not obtained by giving numerical values to the parameters in the parametric solution. All these pseudo-magic squares provide new counterexamples to Guy's aforementioned question. Finally in Section 3.2 we give another elementary method of generating infinitely many pseudo-magic squares of perfect squares, in parametric form, in which the magic sum is always a perfect square.

It is well-known that the magic sum of a 3 x 3 magic square is three times the central number. It follows that the solutions of Section 3.2 can never generate a magic square of perfect squares, but the possibility of a magic square of perfect squares being generated from solutions of the type obtained in Section 3.1 cannot be ruled out. However, no magic square of perfect squares could be found so far, and the problem of LaBar remains an open problem.

2. SEMIMAGIC SQUARES OF SQUARES

Euler was apparently the first to discover the following semimagic square of perfect squares (cf. Dickson [5, p.530]):

$$\begin{bmatrix} (p^2+q^2-r^2-s^2)^2 & (2qr+2ps)^2 & (2qs-2pr)^2 \\ (2qr-2ps)^2 & (p^2-q^2+r^2-s^2)^2 & (2pq+2rs)^2 \\ (2qs+2pr)^2 & (2rs-2pq)^2 & (p^2-q^2-r^2+s^2)^2 \end{bmatrix} \quad (1)$$

This square does not represent all semimagic squares of squares as its magic sum is $(p^2 + q^2 + r^2 + S^2)^2$ whereas there exist semimagic squares of squares whose magic sum is not a perfect square. We will obtain such counterexamples later in this paper.

We will now obtain all semimagic squares whose entries are squares of rational' numbers. It suffices to find rational solutions as all the entries of such squares can be multiplied by a suitable perfect square to obtain seinimagic squares of integer squares.

Lemma: A necessary and sufficient condition that the square of squares

$$\begin{bmatrix} x_3^2 & z_2^2 & y_1^2 \\ z_1^2 & y_3^2 & x_2^2 \\ y_2^2 & x_1^2 & z_3^2 \end{bmatrix} \quad (2)$$

is a semimagic square is that the following equations have a rational solution:

$$x_1^2 - x_2^2 = y_1^2 - y_2^2 = z_1^2 - z_2^2, \quad (3)$$
$$x_1^2 - x_3^2 = y_1^2 - y_3^2 = z_1^2 - z_3^2. \quad (4)$$

Proof: If the equations (3) and (4) are satisfied, by writing $x_1^2 - x_2^2$ = a and $x_1^2 - x_3^2$ = b, we easily find that the sum of each of the rows and columns of the square (2) is $x_1^2 + y_1^2 + z_1^2 - a - b$. Hence (2) is a semimagic square which shows that the condition stated in the lemma is sufficient. To show that the condition is necessary, we denote by R_i and C_i, I = 1, 2, 3, the sums of the entries in the i^{th} row and column respectively. The conditions $R_3 = C_3$ and $R_1 = C_1$ yield the equation (3) while the conditions $R_1 = C_2$ and $R_2 = C_3$ yield the condition (4). This completes the proof.

Theorem : All semimagic squares with entries that are squares of rational numbers are given by (2) where

$$\begin{aligned}
x_1 &= k(p^2qrs^2 - pq^2r^2s - p^2qr + pq^2s + pr^2s - qrs^2 - ps + qr),\\
x_2 &= k(-p^2qrs^2 + pq^2r^2s + p^2qr - pq^2s - 2pqr^2\\
&\quad +2pqs^2 + pr^2s - qrs^2 - ps + qr),\\
x_3 &= k(-p^2qrs^2 + pq^2r^2s - p^2qr + 2p^2rs + pq^2s\\
&\quad -pr^2s - 2q^2rs + qrs^2 - ps + qr),\\
y_1 &= k(-p^2qr^2 + p^2qs^2 + p^2r^2s - q^2r^2s - p^2s + q^2s + qr^2 - qs^2),\\
y_2 &= k(-p^2qr^2 + p^2qs^2 + p^2r^2s - 2pqrs^2 + q^2r^2s\\
&\quad -p^2s + 2pqr - q^2s - qr^2 + qs^2),\\
y_3 &= k(-p^2qr^2 - p^2qs^2 + p^2r^2s + 2pq^2rs\\
&\quad -q^2r^2s + p^2s - 2prs - q^2s + qr^2 + qs^2,\\
z_1 &= k(p^2rs^2 - pq^2r^2 + pq^2s^2 - q^2rs^2 - p^2r + pr^2 - ps^2 + q^2r),\\
z_2 &= k(-p^2rs^2 - pq^2r^2 + pq^2s^2 + 2pqr^2s - q^2rs^2\\
&\quad +p^2r - 2pqs - pr^2 + ps^2 + q^2r),\\
z_3 &= k(-2p^2qrs + p^2rs^2 + pq^2r^2 + pq^2s^2 - q^2rs^2\\
&\quad +p^2r - pr^2 - ps^2 - q^2r + 2qrs),
\end{aligned} \tag{5}$$

where p, q, r, a and k are arbitrary rational parameter.

Proof : It is easily seen that equation (3) will have a rational solution if and only if there exist rational numbers p and q such that

$$x_1 - x_2 = p(y_1 - y_2), \quad p(x_1 + x_2) = y_1 + y_2, \tag{5}$$

$$x_1 - x_2 = q(z_1 - z_2), \quad q(x_1 + x_2) = z_1 + z_2. \tag{6}$$

Similarly, (4) is equivalent to the equations

$$x_1 - x_3 = r(y_1 - y_3), \quad r(x_1 + x_3) = y_1 + y_3, \tag{7}$$

$$x_1 - x_3 = s(z_1 - z_3), \quad s(x_1 + x_3) = z_1 + z_3, \tag{8}$$

where r and s are rational numbers. Equations (6), (7) , (8), (9) are 8 linear equations in the 9 variables x_i, x_2, x_3, y_1, u_2, y_3, z_1, z_2, z_3, and hence are readily solved to give the solution (5) and, in view of the lemma, all semimagic squares of squares are now given by (2).

3. PSEUDO–MAGIC SQUARES OF SQUARES

We will now give elementary methods of generating infinitely many pseudo- magic squares in which the entries are perfect squares of rational numbers. As in the case of semimagie squares, it suffices to find rational solutions as these easily lead to pseudo-magic squares of integer squares by multiplying by a suitable integer square. Further, while the definition of pseudo-magic squares requires that the entries in either of the diagonals should add up to the magic sum, in all the pseudo-magic squares considered in this paper, it is the principal diagonal that adds up to the magic sum.

3.1 Pseudo-magic squares of squares in which the magic sum is not a perfect square

Lemma: A necessary and sufficient condition that the square of squares (2) is a pseudo-magic square is that the following equations have a rational solution:

$$x_1^2 - x_2^2 = y_1^2 - y_2^2 = z_1^2 - z_2^2, \quad (9)$$

$$x_1^2 + x_2^2 - 2x_3^2 = y_1^2 + y_2^2 - 2y_3^2 = z_1^2 + z_2^2 - 2z_3^2 = 0. \quad (10)$$

Proof: If (10) and (11) are satisfied, we get $x_1^2 + x_2^2 = 2x_3^2$, and writing, $x_1^2 - x_2^2 = 2a$ we get $x_1^2 = x_3^2 + a, x_2^2 = x_2^3 - a$. Similarly, we get $y_1^2 = y_3^2 + a,\ y_2^2 = y_3^2 - a$, and $z_1^2 = z_3^2 + a,\ z_2^2 = z_2^3 - a$, and it is now readily seen that the sum of each row, column and the principal digonal of the square (2) is $y_1^2 = y_3^2 + z_2^3$. Thus the condition stated in the leibma is sufficient. To show that it is also necessary, we note that since every pseudo-magic square is necessarily a semimagie square, (3) (which is same as equation (10)) and (4) must be satisfied, If the square (2) is a pseudo-magic square then, in addition to (3) and (4), the following condition obtained by equating the sum of the entries of the principal diagonal to the sum of the entries in the first row, is also satisfied:

Combining the last part of (4) with (12), we get

$$z_1^2 + z_2^2 - 2z_3^2 = 0. \quad (13)$$

Further, on subtracting two times (4) from (3), we get

$$x_1^2 + x_2^2 - 2x_3^2 = y_1^2 + y_2^2 - 2y_3^2 = z_1^2 + z_2^2 - 2z_3^2. \quad (14)$$

Now (11) follows from (13) and (14) and the proof is complete. Equations (10) and (11) can be solved in various ways leading to pseudo-magic squares in which the magic sum is either a perfect square or not a perfect square. In this section we will consider solutions of these equations such that the magic sum of the resulting pseudo-magic square is not a perfect square.

We may consider equation (11) as three independent quadratic equations of the type $x_1^2 + x_2^2 - 2x_3^2 = 0$, and hence obtain its complete solution which may be written as follows:

$$\begin{aligned}
x_1 &= k_1(X_1^2 - 2X_1X_2 - X_2^2), \\
x_2 &= k_1(X_1^2 + 2X_1X_2 - X_2^2), \\
x_3 &= k_1(X_1^2 + X_2^2), \\
y_1 &= k_2(Y_1^2 - 2Y_1Y_2 - Y_2^2), \\
y_2 &= k_2(Y_1^2 + 2Y_1Y_2 - Y_2^2), \\
y_3 &= k_2(Y_1^2 + Y_2^2), \\
z_1 &= k_3(Z_1^2 - 2Z_1Z_2 - Z_2^2), \\
z_2 &= k_3(Z_1^2 + 2Z_1Z_2 - Z_2^2), \\
z_3 &= k_3(Z_1^2 + Z_2^2),
\end{aligned} \quad (15)$$

where X_i, Y_i, Z_i, I = 1,2 as well as k1, k2, k3 are arbitrary parameters. Substituting these values of X_i, Y_i, Z_i, i = 1,2 in (10), we get

$$k_1^2 X_1 X_2 (X_1^2 - X_2^2) = k_2^2 Y_1 Y_2 (Y_1^2 - Y_2^2) = k_3^2 Z_1 Z_2 (Z_1^2 - Z_2^2). \tag{16}$$

We write $Y_1 = m X_1$, $Y_2 = -m(X_1 + X_2)$, when the first part of equation (16) reduces to a linear equation in X_1 and X_2, and we get the following solution for this part of equation (16):

$$\begin{aligned} X_1 &= (k_1^2 + k_2^2 m^4), & X_2 &= (k_1^2 - 2k_2^2 m^4), \\ Y_1 &= m(k_1^2 + k_2^2 m^4), & Y_2 &= -m(2k_1^2 - k_2^2 m^4), \end{aligned} \tag{17}$$

where m is an arbitrary parameter. The remaining part of equation (16) may now be written as

$$\begin{aligned} &3m^4(k_1^2 - 2m^4k_2^2)(2k_1^2 - m^4k_2^2)(k_1^2 + m^4k_2^2)k_1^2k_2^2 \\ &= k_3^2 Z_1 Z_2 (Z_1 - Z_2)(Z_1 + Z_2). \end{aligned} \tag{18}$$

To solve equation (18), we note that k_3 will be rational if Z_1, Z_2 are so chosen that

$$3(k_1^2 - 2m^4k_2^2)(2k_1^2 - m^4k_2^2)(k_1^2 + m^4k_2^2) Z_1 Z_2 (Z_1 - Z_2)(Z_1 + Z_2), \tag{19}$$

is a perfect square. Further, it follows from (16) that a trivial solution of (18) is given by $Z_1 = X_1$, $Z_2 = X_2$, $k_3 = k_1$. Since (19) is a quartic form in Z_1, Z_2, we can easily find the desired values of Z_1, Z_2, by using the known trivial solution and following the usual methods described in [5, p. 639]. Substituting the values of X_i, Y_i, Z_i, i = 1,2, in (15), we get a solution of equations (10) and (11) and hence a pseudo-magic square in parametric form. We note that while we have apparently three parameters in the final solution, two of the parameters are not independent. Writing the polynomial $c_o t^n + c_1 t^{n-1} + \ldots + C_n$ briefly as $(c_0, c_1, \ldots, c_n)$, this pseudo-magic square may be written as (2) where x, y, z are polynomials in terms of a single variable t defined as follows:

x1 = (–32,0,224,0, –128,0, –1648,0,3760,0, –5944,0,3136,0, 1052,0,70,0),
x2 = (32,0, –32,0, –544,0,496,0,800,0, –3896,0,6728,0, –3164,0, –490,0),
x3 = (32,0, –128,0, –64,0,640,0, –2560,0,3904,0, –4144,0,2320,0,350,0),
x4 = (16,0,80,0, –536,0, –544,0, 2140,0, –5428,0,5326,0, –424,0, –140),
y2 = (–112,0,400,0,680,0, –2816,0,5660,0, –5444,0,158,0,1264,0,140),
y3 = (80,0, –272,0, –472,0, 1744,0, –4180,0,4420,0, –1954,0,844,0, 140),
y4 = (16,0, –256,0,352,0, –256,0,3448,0, –5632,0, 1240,0,3488,0, –1151,0),
Z2 = (16,0,128,0, –992, 0, 2624, 0, –392,0, –4576, 0, 2824, 0, –2032, 0, 1249, 0),
Z3 = (16,0, –64,0,832,0, –2272,0,664,0,2384,0,1168,0, –2728,0,1201,0).

It is easily verified that the magic sum of this pseudo-magic square is not a perfect square. As a numerical example, if we take t = 2 in the above solution, we get the following pseudo-magic square

$$\begin{bmatrix} 68962^2 & 1414561^2 & 54694^2 \\ 1411199^2 & 87986^2 & 97498^2 \\ 111766^2 & 2378^2 & 1412881^2 \end{bmatrix}$$

whose magic sum, expressed as a product of prime numbers, may be written as 3.7.37.15077.171469 and hence is not a perfect square.

We can find infinitely many values of Z_1, Z_2 that make (19) a perfect square and hence obtain more such pseudo-magic squares in parametric form. The next pseudo-magic square thus obtained has polynomials of degree 74 in a single variable as its entries. Its magic sum is again not a perfect square.

We present below another pseudo-magic square, obtained by solving equations (10) and (11) in a different way:

$$\begin{bmatrix} \{(x^4+22x^2+9)z\}^2 & \{(x^4+4x^3-10x^2-12x+9)y\}^2 & \{2(x^4+4x^2+12x-9)z\}^2 \\ \{(x^4-4x^3-10x^2+12x+9)y\}^2 & \{2(x^4+2x^3+2x^2-6x+9)z\}^2 & \{(x^4+8x^3-10x^2+24x+9)z\}^2 \\ \{2(x^4+4x^3-4x^2-9)z\}^2 & \{(x^4-8x^3-10x^2-24x+9)z\}^2 & \{(x^4-2x^2+9)y\}^2 \end{bmatrix}$$

where $z = x^2 - 3$ and x, y are related by the quartic equation

$$y^2 = 2x^4 - 18. \tag{20}$$

It is readily verified that the above indeed gives a pseudo—magic square. Now (20) is a quartic model of an elliptic curve of rank 1 and accordingly infinitely many rational solutions of this equation may be obtained, one solution being x = 99/47, y = 10212/2209 which leads to the pseudo-magic square

$$\begin{bmatrix} 98434526^2 & 8968837^2 & 83721863^2 \\ 69684283^2 & 67982503^2 & 85449554^2 \\ 47264057^2 & 109895794^2 & 49680677^2 \end{bmatrix}$$

whose magic sum is 2.32,386051.2414641573 which is not a perfect square.

3.2. Pseudo-magic squares of squares in which the magic sum is a perfect square

While we can obtain pseudo-magic squares of squares in which the magic sum is a perfect square by solving equations (10) and (11), we will give a simpler method of obtaining such solutions. The method illustrates why such squares are more easily found as compared to the case in which the magic sum is not a perfect square.

In the semimagic square (1) we interchange the second and third columns to obtain the following semimagic square which naturally also has the magic sum $(p^2 + q^2 + r^2 + s^2)^2$:

$$\begin{bmatrix} (p^2+q^2-r^2-s^2)^2 & (2qs-2pr)^2 & (2qr+2ps)^2 \\ (2qr-2ps)^2 & (2pq+2rs)^2 & (p^2-q^2+r^2-s^2)^2 \\ (2qs+2pr)^2 & (p^2-q^2-r^2+s^2)^2 & (2rs-2pq)^2 \end{bmatrix} \tag{21}$$

The condition that the sum of entries of the principal diagonal of this square is also equal to the magic sum reduces to the following equation:

$$(2q^2 - r^2 - s^2)p^2 - q^2r^2 + 2r^2s^2 - q^2s^2 = 0. \quad (22)$$

Computer trials readily yield a number of numerical solutions of (22) leading to pseudo-magic squares of squares of relatively small integers. To obtain a parametric solution, we substitute p = 2r – q in (22) when this equation reduces to

$$q^2 - 2qr - 2r^2 - s^2 = 0. \quad (23)$$

The complete solution of (23) is readily found and we thus obtain the following pseudo-magic square with entries of degree 8 in the parameter t:

$$\begin{bmatrix} 4(t^4 - 2t^3 + 2t^2 + 6t + 9)^2 & 4(t^4 + 4t^3 - 4t^2 - 9)^2 & (t^4 - 8t^3 - 10t^2 - 24t + 9)^2 \\ 4(t^4 + 4t^2 + 12t - 9)^2 & (t^4 + 22t^2 + 9)^2 & 4(t^4 + 4t^2 - 12t - 9)^2 \\ (t^4 + 8t^3 - 10t^2 + 24t + 9)^2 & 4(t^4 - 4t^3 - 4t^2 - 9)^2 & 4(t^4 + 2t^3 + 2t^2 - 6t + 9)^2 \end{bmatrix}.$$

This square is readily seen to be equivalent to the square with entries of degree 8 given by Bremner in [3].

We now describe a method of obtaining other parametric solutions of (22) using a given solution. If (p_1, q_1, r_1, s_1) is a known solution of (22), we substitute in this equation,

$$p = gx + p_1, \quad q = gx + q_1, \quad r = gx + r_1, \quad s = hx + s_1, \quad (24)$$

where g, h, and x are arbitrary, when (22) reduces to the following quadratic equation in x:

$$\begin{aligned} &2g(g^2 - h^2)(p_1 + q_1 - 2r_1)x^2 + \{(p_1^2 + 8p_1q_1 - 4p_1r_1 + q_1^2 \\ &-4q_1r_1 - 2r_1^2)g^2 - 4(p_1 + q_1 - 2r_1)s_1gh - (p_1^2 + q_1^2 - 2r_1^2)h^2\}x \\ &-(2p_1^2 + 2q_1^2 - 4r_1^2)s_1h + (4p_1^2q_1 - 2p_1^2r_1 + 4p_1q_1^2 - 2p_1r_1^2 \\ &-2p_1s_1^2 - 2q_1^2r_1 - 2q_1r_1^2 - 2q_1s_1^2 + 4r_1s_1^2)g = 0. \end{aligned} \quad (25)$$

We choose

$$\begin{aligned} g &= (2p_1^2 + 2q_1^2 - 4r_1^2)s_1, \\ h &= (4p_1^2q_1 - 2p_1^2r_1 + 4p_1q_1^2 - 2p_1r_1^2 - 2p_1s_1^2 \\ &\quad -2q_1^2r_1 - 2q_1r_1^2 - 2q_1s_1^2 + 4r_1s_1^2), \end{aligned} \quad (26)$$

when the constant term in (25) vanishes and the equation is readily solved to obtain a nonzero value of x which, on using (24), gives a rational solution of (22). The trivial solution $(p_1, q_1. r_1, s_1) = (1, t, -1, 1)$ of equation (22) yields the parametric solution obtained earlier by substituting p = 2r –q in (22). However, another trivial solution of (22) may be obtained by taking p = 0 and solving the resulting simple equation for q, r and s. Using this trivial solution, we can obtain a non-trivial solution of (22) which leads to a pseudo-magic square of squares, in parametric form, with entries of degree 56 in one parameter. Further, using the non-trivial solution of (22) now obtained, we can apply the method described above to find another parametric solution of (22) and this process may be continued. We may thus obtain infinitely many pseudo-magic squares of squares in parametric form.

Equation (22) may also be solved by noting that a rational solution for p will exist if $(2q^2 - r^2 - s^2)(q^2r^2 - 2r^2s^2 + q^2s^2)$ could be made a perfect square. This may be considered as a quartic in r and a trivial solution is given by r = q.

CONCLUSION

we can find infinitely many values of r that will make the above quartic a perfect square. This leads to new parametric solutions of equation (22) and hence we obtain more pseudo-magic squares of squares, in parametric form, the first such pseudo-magic square having entries of degree 20 in a single parameter.

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